

Rationality, quasirationality and finite W -algebras

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Abstract

Some of the consequences that follow from the C_2 condition of Zhu are analysed. In particular it is shown that every conformal field theory satisfying the C_2 condition has only finitely many n -point functions, and this result is used to prove a version of a conjecture of Nahm, namely that every representation of such a conformal field theory is quasirational. We also show that every such vertex operator algebra is a finite W -algebra, and we give a direct proof of the convergence of its characters as well as the finiteness of the fusion rules.

1 Introduction

Conformal field theory has had a major impact on modern theoretical physics as well as modern mathematics. From the point of view of physics, conformal field theory plays a central rôle in string theory, at present the most promising candidate for a unifying theory of all forces. On the other hand, conformal field theory inspired the purely mathematical definition of vertex operator algebras, which has led to beautiful and deep results in the theory of finite groups and number theory.

The significance and power of conformal field theory was first conclusively demonstrated by the work of Belavin, Polyakov and Zamolodchikov [1]. They fixed a general framework for its study which was further developed by Moore and Seiberg [27], in particular. On the other hand, the mathematical theory of vertex operator algebras is due to Borcherds [2, 3] and Frenkel, Lepowsky & Meurman [12], and was further developed by Frenkel, Huang & Lepowsky [10], Zhu [33], Kac [21] and others. Apart from this algebraic viewpoint, there exists also a geometrical approach that was directly inspired by string theory (in particular the work of Friedan & Shenker [13]) and that has been put on a mathematical foundation by Segal [31] and Huang [18, 19, 20].

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Much has been learned about conformal field theory, but there are still a number of conceptual problems that have not been resolved so far. One of them concerns the question of how to characterise the class of ‘rational’ theories, *i.e.* those theories that are in some sense finite and tractable. Various definitions of rationality have been proposed, but the interrelations between the different assumptions are not very well understood.

One of the assumptions that were introduced by Zhu in [33] in order to be able to prove the convergence of the characters is the condition (sometimes referred to as the C_2 condition) that a certain quotient space of the vertex operator algebra is finite-dimensional. This is a slightly technical assumption; however, it has the great virtue of being easily testable in concrete examples. In this paper we analyse the consequences that follow from this condition. As we shall show, the C_2 condition implies that a whole family of quotient spaces are finite-dimensional, and this in turn is sufficient to prove that the theory has only finitely many n -point functions (Theorem 11), and in particular that all fusion rules between irreducible highest weight representations are finite (Corollary 14). We also prove that the C_2 condition implies that each highest weight representation of the theory is quasirational (Theorem 13); this proves a version of a conjecture of Nahm [28]. Finally, we show that every such vertex operator algebra is a finite W -algebra, and we give a direct proof of the convergence of its characters (see also [7, 23] for independent proofs of these results.) Most of these results hinge on finding a small spanning set of the vacuum representation (Proposition 8); we expect that this result may also be useful in other contexts and applications.

If we assume in addition that Zhu’s algebra is semisimple, we can also find an upper bound on the (effective) central charge of the theory in terms of the dimension of the C_2 quotient space (Proposition 15).

The paper is organised as follows. In Section 2 we fix our conventions and introduce some notation. In Section 3 we define a class of quotient spaces that generalise the construction of the C_2 space (and of Zhu’s algebra), and we prove a number of simple properties. In Section 4 we recall the definition of a highest weight representation and we explain in which sense Zhu’s algebra classifies these representations. Section 5 is concerned with the different definitions of rationality. The central Proposition is proven in Section 6, and a few simple consequences are derived. In Section 7 we use this result to prove that each conformal field theory satisfying the C_2 condition has only finitely many n -point functions, and we show that this implies Nahm’s conjecture. Section 8 describes our bound on the central charge. Section 9 gives a more precise description of one of the quotient spaces which figure prominently in the proofs of the preceding results, and Section 10 contains some conclusions and outlook for further work. Finally, we have included an appendix where two quite technical calculations are described in detail.

2 Notation

We assume the reader is familiar with basic notions of conformal field theory, as found for instance in [5, 14]. Some acquaintance with the language of vertex operator algebras [2, 12, 21] is also helpful.

At all times in this paper we are considering a fixed chiral bosonic conformal field theory defined on the sphere \mathbb{P} . To be precise, by a “chiral bosonic conformal field theory” we mean an object of the type discussed in [15]. From this point of view, a conformal field theory on \mathbb{P} is defined in terms of its amplitudes. We assume that the amplitudes are local, Möbius invariant, and satisfy the cluster decomposition property (that guarantees that the spectrum of the scaling operator L_0 is bounded by zero from below, with a unique state Ω of eigenvalue $h = 0$). We also assume that the theory is conformal, *i.e.* that it possesses a stress energy tensor $V(L, z) = L(z)$ whose modes

L_n satisfy the Virasoro algebra. The details of the chosen formalism are not essential to following the ideas of this paper, and indeed, the whole argument could be rewritten in terms of the standard axioms of vertex operator algebras [21].

The amplitudes that define the theory are written as $\langle \prod_{a=1}^k V(\psi_a, z_a) \rangle$, where the vertex operator corresponding to $\psi \in V$ is denoted by $V(\psi, z)$. The vector space V consists of quasiprimary states that generate the whole theory; for convenience we shall occasionally assume that V is the space of all quasiprimary states. We sometimes write $V(\psi, z)$ in terms of modes as

$$V(\psi, z) = \sum_{n \in \mathbb{Z}} V_n(\psi) z^{-n-h_\psi} = \sum_{m \in \mathbb{Z}} V_{(m)}(\psi) z^{-m-1}, \quad (1)$$

where $V_{(m)}(\psi) = V_{m+1-h_\psi}(\psi)$ is the moding that is commonly used in the mathematical literature.

We will frequently consider meromorphic functions and differentials defined on the Riemann sphere \mathbb{P} . It is convenient to use the language of “divisors” (see [17]) to classify the zeros and poles of these functions. We now give a brief review of the facts relevant for us. A divisor on \mathbb{P} is, by definition, any formal sum of the form

$$D = \sum_{P \in \mathbb{P}} c_P [P], \quad c_P \in \mathbb{Z}, \quad \text{finitely many } c_P \neq 0. \quad (2)$$

Divisors can be added and subtracted in the obvious way, and we say $D \geq 0$ if all $c_P \geq 0$. Now let $\nu_P(f)$ denote the order of vanishing of f at P (so $\nu_P(f)$ is negative if f has a pole at P), and define the divisor of f to be

$$\text{div } f = \sum_{P \in \mathbb{P}} \nu_P(f) [P]. \quad (3)$$

Clearly $\text{div } fg = \text{div } f + \text{div } g$, and $\text{div } f \geq 0$ just if f is holomorphic (*i.e.* constant).

We can similarly define $\text{div } \omega$ where ω is a meromorphic k -differential on \mathbb{P} ; explicitly, such an ω can always be written as $\omega = f dz^{\otimes k}$ for some f , and then we have

$$\text{div } \omega = \text{div } f + k \text{div } dz = \text{div } f - 2k[\infty]. \quad (4)$$

(This definition expresses the fact that dz has a pole of order 2 at infinity.)

The crucial analytic property which the amplitudes of the theory must possess by definition [15] is that, for any $\psi \in V$, $\langle V(\psi, z) \prod_{i=1}^k V(\psi_i, z_i) \rangle dz^{\otimes h_\psi}$ depends meromorphically on $z \in \mathbb{P}$ and has poles only for $z = z_i$.

The Fock space \mathcal{H} of the theory is spanned by finite linear combinations of states of the form

$$\Psi = V_{(n_1)}(\psi_1) \cdots V_{(n_k)}(\psi_k) \Omega, \quad (5)$$

where $\psi_i \in V$, Ω denotes the unique (vacuum) state with conformal weight $h = 0$, and $n_i \in \mathbb{Z}$. Any product of vertex operators $V(\phi_1, u_1) \cdots V(\phi_l, u_l)$ defines a linear functional on the Fock space by

$$\eta_{V(\phi_1, u_1) \cdots V(\phi_l, u_l)}(\Psi) = \oint_0 dz_1 z_1^{n_1} \cdots \oint_0 dz_k z_k^{n_k} \langle V(\phi_1, u_1) \cdots V(\phi_l, u_l) \prod_{i=1}^k V(\psi_i, z_i) \rangle, \quad (6)$$

where the contours are chosen so that $|z_1| > |z_2| > \cdots > |z_k|$. The Fock space is the space spanned by vectors of the form (5), modulo states that vanish in each linear functional associated to any product of vertex operators. In (6) we have considered the Fock space at $0 \in \mathbb{P}$; however, since the amplitudes are translation invariant, it is clear that one can similarly consider the Fock space at any other point on the Riemann sphere.

3 The subspaces $A_{\mathbf{u}}$

We begin by introducing a generalization of the quotients of \mathcal{H} which appeared in [15, 29, 33]. For fixed $\mathbf{u} = (u_1, \dots, u_k) \in (\mathbb{P} - \{0\})^k$ (where we do not require that the u_i be distinct), we define

$$O_{\mathbf{u}} = \text{Span} \left\{ \oint_0 dz g(z) V(\psi, z) \chi \mid \chi \in \mathcal{H}, \psi \in V, g \text{ meromorphic}, \right. \\ \left. \text{div } g dz^{\otimes -h_\psi+1} \geq -N[0] + \sum_{i=1}^k h_\psi[u_i] \text{ for some } N \geq 0 \right\}. \quad (7)$$

We then set

$$A_{\mathbf{u}} = \mathcal{H} / O_{\mathbf{u}}. \quad (8)$$

Because of the Möbius invariance of the amplitudes, we may assume that one of the u_i , u_1 say, is equal to ∞ . If none of the other u_j are equal to ∞ , one can give an explicit description of $O_{\mathbf{u}}$ as the space spanned by the states of the form $V_{\mathbf{u}}^{(M)}(\psi)\chi$ with $M > 0$, where

$$V_{\mathbf{u}}^{(M)}(\psi) = \oint_0 \frac{d\zeta}{\zeta^{M+1}} V \left[\left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{L_0} \psi, \zeta \right] \quad (9)$$

and $\psi \in V$, $\chi \in \mathcal{H}$. For the case where the u_i are distinct, this space has been considered before in [15], where it was denoted by A_k (but we now renounce that notation in favour of one described below.) In the case where all k of the u_i equal ∞ , one can give a similarly explicit description: in this case $O_{\mathbf{u}}$ is simply spanned by states of the form $V_{-N-(k-1)h_\psi}(\psi)\chi$ with $N > 0$. This choice of \mathbf{u} is particularly convenient since the resulting $O_{\mathbf{u}}$ is spanned by states of definite conformal weight; this makes calculations significantly simpler.

The original motivation for the definition of $A_{\mathbf{u}}$, in the case where all u_i are distinct, stemmed from the fact that the algebraic dual space $A_{\mathbf{u}}^*$ describes the correlation functions involving k highest weight states at u_1, \dots, u_k (this was first observed by Zhu in [33] for $\mathbf{u} = (-1, \infty)$). More generally, one finds

Theorem 1. *There is a one-to-one linear correspondence between elements $\eta \in A_{\mathbf{u}}^*$ and systems of correlation functions on the sphere, i.e. maps*

$$((\psi_1, z_1), \dots, (\psi_l, z_l)), \psi_i \in V, z_i \in \mathbb{P} \mapsto \left\langle \prod_{j=1}^l V(\psi_j, z_j) \right\rangle_\eta \in \mathbb{C} \quad (10)$$

such that the $\langle \prod_{j=1}^l V(\psi_j, z_j) \rangle_\eta$ (regarded only as functions of the z_j) obey the operator product relations of the theory defined on the sphere (see [15] for a precise definition), and have the “highest weight” property

$$\text{div} \langle V(\psi, z) \rangle_\eta dz^{\otimes h_\psi} \geq - \sum_{i=1}^k h_\psi[u_i]. \quad (11)$$

Proof. Given any system of correlation functions one can construct a linear functional η on \mathcal{H} by contour integration, as discussed in Section 2. If we further require (11), then this functional vanishes on $O_{\mathbf{u}} \subset \mathcal{H}$, and therefore $\eta \in A_{\mathbf{u}}^*$. Conversely, any $\eta \in A_{\mathbf{u}}^*$ defines formal Laurent series, whose convergence to functions with the required analytic properties was proven in [29]. (Strictly

speaking the proof was only given under the additional hypothesis that the u_i be distinct; however, that hypothesis is actually not required anywhere in the proof.) \blacksquare

It is often convenient to use a shorthand notation where we only keep track of the number of coincident points u_i . Let us thus define $A_{\mathbf{n}}$ where \mathbf{n} is a multi-index $\mathbf{n} = [n_1, \dots, n_l]$; this denotes the space $A_{\mathbf{u}}$ for the case where n_1 of the u_i are equal to v_1 , n_2 of the u_i are equal to $v_2 \neq v_1$, etc. We define $X_{\mathbf{n}}$ to be the corresponding configuration space, namely the set of all $\mathbf{u} \in \mathbb{P}^{|\mathbf{n}|}$ (where $|\mathbf{n}| = n_1 + n_2 + \dots + n_l$) for which the first n_1 coordinates are coincident, the second n_2 coordinates are coincident, and so on. The usefulness of this notation depends on the following fact:

Theorem 2. *Suppose $A_{(\infty^k)}$ is finite-dimensional and let $|\mathbf{n}| = k$. Then the space $A_{\mathbf{n}}$ is independent of the choice of $\mathbf{u} \in X_{\mathbf{n}}$, in the sense that choosing a homotopy class of paths from \mathbf{u} to \mathbf{u}' in $X_{\mathbf{n}}$ determines a natural isomorphism $A_{\mathbf{u}} \simeq A_{\mathbf{u}'}$.*

Proof. Using Theorem 1 we can regard $A_{\mathbf{n}}^*$ as a space of correlation functions. First consider the case $\mathbf{n} = (1, 1, \dots, 1)$ where all u_i are distinct. In that case we can introduce a more suggestive notation for the correlation functions, namely, we write

$$\left\langle \prod_{i=1}^k W(\phi_i, u_i) \prod_{i=1}^k V(\psi_i, z_i) \right\rangle \equiv \left\langle \prod_{i=1}^k V(\psi_i, z_i) \right\rangle_{\eta}. \quad (12)$$

(Here the formal symbols $W(\phi_i, u_i)$ represent insertions of highest weight states.) Given all correlation functions at some fixed \mathbf{u} , the Knizhnik-Zamolodchikov equation determines them at all \mathbf{u} using the fact that the Virasoro algebra acts geometrically; more specifically, if $u_1 \neq \infty$ then

$$\frac{\partial}{\partial u_1} \left\langle \prod_{i=1}^k W(\phi_i, u_i) \right\rangle = \oint_{u_1} dz \left\langle \prod_{i=1}^k W(\phi_i, u_i) L(z) \right\rangle. \quad (13)$$

Using this formula systematically one can construct a family of differential equations, to be solved in the space obtained by gluing together the $A_{\mathbf{u}}^*$ at different points \mathbf{u} ; if these equations admit solutions, we then expect that they will define the analytic continuation from correlation functions at \mathbf{u} to correlation functions at \mathbf{u}' , proving the theorem.

To prove that solutions actually exist one has to impose the condition that $A_{(\infty^k)}$ is finite-dimensional (specifically, what one uses is the fact that a basis for $A_{(\infty^k)}$ corresponds to a spanning set for each $A_{\mathbf{u}}$, $\mathbf{u} \in X_{\mathbf{n}}$.) Under this assumption it is shown in [29] that the $A_{\mathbf{u}}^*$ (and hence the $A_{\mathbf{u}}$) indeed fit together to form a vector bundle over $X_{\mathbf{n}}$ which possesses a natural flat connection given by (13). The argument given there extends straightforwardly to the case where the u_i need not be distinct. \blacksquare

The space $O_{(\infty, \infty)}$ is the C_2 space of Zhu, so if $A_{[2]}$ is finite-dimensional, the C_2 condition of Zhu is satisfied. On the other hand, $A_{[1,1]}$ is isomorphic to Zhu's algebra (compare also Section 4).

The space $A_{[k]}$ has been considered before in [29], where it was denoted by \mathcal{H}/C_k . As we now show, its dimension provides an upper bound on the dimension of the spaces $A_{\mathbf{n}}$ with $|\mathbf{n}| = k$; this result was already used in [33] for the special case $\mathbf{n} = (1, 1)$ (see also [29]).

Lemma 3. $\dim A_{\mathbf{n}} \leq \dim A_{[|\mathbf{n}|]}$.

Proof. Fix $\mathbf{u} = (\infty, u_2, \dots, u_k)$ (by Möbius invariance this involves no loss of generality). It follows from (7) that $O_{\mathbf{u}}$ is generated by the states of the form

$$V_{\mathbf{u}}^{(N)}(\psi)\chi = \sum_{s=0}^{(k-1)h_{\psi}} c_s V_{-N-(k-1)h_{\psi}+s}(\psi)\chi \quad (14)$$

where $N > 0$ and c_s are some constants (depending on \mathbf{u}) with $c_0 = 1$. On the other hand, $O_{(\infty^k)}$ is generated by the states of the form $V_{-N-(k-1)h_\psi}(\psi)\chi$ with $N > 0$.

Let $\{\phi_1, \dots, \phi_M\}$ be a set of representatives for \mathcal{H} modulo $O_{(\infty^k)}$. We claim that these vectors also span \mathcal{H} modulo $O_{\mathbf{u}}$. Suppose that this is not the case, and let Ψ be a vector of minimal conformal weight that does not differ by an element in $O_{\mathbf{u}}$ from a linear combination of ϕ_1, \dots, ϕ_M . By assumption we can write

$$\Psi = \sum_{j=1}^M b_j \phi_j + \sum_{r=1}^L V_{-N_r-(k-1)h_r}(\psi_r)\chi_r. \quad (15)$$

But then

$$\hat{\Psi} = \Psi - \sum_{j=1}^M b_j \phi_j - \sum_{r=1}^L V_{\mathbf{u}}^{(N_r)}(\psi_r)\chi_r \quad (16)$$

is a linear combination of vectors whose conformal weight is strictly smaller than that of Ψ . By the minimality of Ψ it then follows that $\hat{\Psi}$ differs by an element in $O_{\mathbf{u}}$ from a linear combination of ϕ_1, \dots, ϕ_M , and we have the desired contradiction. ■

It should be noted that the dimension can actually decrease when we ‘split points’. The simplest example for this phenomenon occurs already for $\mathbf{n} = [1, 1]$: the e_8 level 1 theory is self-dual (*i.e.* the only representation is the vacuum representation), and therefore has no nontrivial two-point functions, implying $\dim A_{[1,1]} = 1$; on the other hand, it is easy to see by inspection that $\dim A_{[2]} \geq 249$.

4 Representations and Zhu’s algebra

We now shift from considering the vacuum representation \mathcal{H} to more general representations of the conformal field theory.

A representation of the conformal field theory is defined in terms of the amplitudes it induces [15, 33],

$$\langle W(\phi_1, u_1) W(\phi_2, u_2) \prod_{i=1}^k V(\psi_i, z_i) \rangle, \quad (17)$$

where the $\psi_i \in V$ are arbitrary. The amplitudes have the crucial property that they respect the operator product relations of the meromorphic conformal field theory. Furthermore, the amplitudes are Möbius covariant, and are analytic as a function of the z_i , except for possible poles at $z_i = z_j$, $i \neq j$, and singularities at $z_i = u_j$. We call the representation amplitudes *non-singular* if the singularities at $z_i = u_j$ are poles of finite order; a non-singular representation amplitude is *highest weight* if the order of the pole at $z_i = u_j$ is bounded by h_{ψ_i} .

We can construct from the amplitudes two vector spaces \mathcal{H}^1 and \mathcal{H}^2 which form modules for the modes $V_n(\psi)$ for $\psi \in \mathcal{H}$. These modules are generated by the action of the modes (defined via contour integrals around the u_i) from ϕ_1 and ϕ_2 , respectively. The actual module is then a quotient space of the space so obtained, where we remove “null vectors” by identifying states whose difference vanishes in all amplitudes (17).

The requirement that the amplitudes respect the operator product relations implies that the action of the modes satisfies the “Jacobi identity” required in algebraic definitions of representation

(given *e.g.* in [21]). If the representation amplitudes are non-singular, then the two representations are “weak modules” in the sense of [6], *i.e.* \mathcal{H}^i has the property that, for any $\psi \in \mathcal{H}$ and $\chi \in \mathcal{H}^i$, $V_n(\psi)\chi = 0$ for $n > N$ (where N may depend on ψ, χ). Finally, if the representation amplitudes are highest weight, then the two representations are highest weight representations, *i.e.* \mathcal{H}^i is generated from a single state ϕ_i with the property that $V_n(\psi)\phi_i = 0$ whenever $n > 0$.

On the other hand, we can construct representation amplitudes (that have the appropriate analytic properties) from purely algebraic data. Indeed, it follows from Theorem 1 that each element in the algebraic dual of $A_{(u_1, u_2)}$ defines representation amplitudes that have the highest weight property. It was furthermore shown by Zhu [33] (see also [14, 15] for an exposition more in line with the present point of view) that $A_{(u_1, u_2)}$ has the structure of an algebra, and that the equivalence classes of representation amplitudes (where we identify amplitudes that define equivalent modules \mathcal{H}^i) are in one-to-one correspondence with representations of $A_{(u_1, u_2)}$. Finally, the irreducible representations R of Zhu’s algebra are in one-to-one correspondence with the irreducible highest weight representations \mathcal{H}^R of the conformal field theory.

The algebra structure of Zhu’s algebra is most easily understood for $\mathcal{A} = A_{(\infty, -1)}$, whence it is defined by

$$\psi * \chi \equiv V_{(\infty, -1)}^{(0)}(\psi)\chi, \quad (18)$$

where $V_{(\infty, -1)}^{(0)}(\psi)$ is given in (9). This product is characterized by the identity (emphasized by Brungs and Nahm in [4])

$$V_0(\psi * \chi) = V_0(\psi)V_0(\chi) \quad (19)$$

which holds when both sides act on highest weight states, so that \mathcal{A} is essentially the algebra of zero modes of fields in the vacuum sector acting on highest weight states.

Theorem 1 states that $A_{(u_1, \dots, u_k)}^*$ describes the space of correlation functions that correspond to k highest weight states. If we are, however, interested in understanding the different ways in which the various representations of the theory can couple in k -point functions, then this description contains a certain redundancy. In particular, we can act with zero modes $V_0(\psi)$ on any of the ϕ_i in (12), and this will produce another highest weight state in the same representation. It is therefore useful to study $A_{\mathbf{u}}^*$ as a representation of k copies of the zero mode algebra \mathcal{A} acting at the k points u_i .

Theorem 4. *Fix a multi-index \mathbf{n} . For any i with $n_i = 1$ there is a natural map of algebras, $\rho_i : \mathcal{A} \rightarrow \text{End}(A_{\mathbf{n}})$. The dual map $\rho_i^* : \mathcal{A} \rightarrow \text{End}(A_{\mathbf{n}}^*)$ satisfies the identity (for $u_i \neq \infty$)*

$$\langle \cdots \rangle_{\rho_i^*(\psi)\eta} = \oint_{u_i} dz (z - u_i)^{h_\psi - 1} \langle V(\psi, z) \cdots \rangle_\eta, \quad (20)$$

i.e. it is the action of zero modes.

Proof. Without loss of generality we may assume that $\mathbf{u} = (\infty, u_2, \dots, u_k)$. Using the notation introduced in (9) we define $\rho_1(\psi) = V_{\mathbf{u}}^{(0)}(\psi)$. It is shown in the appendix that for $L > 0$

$$[V_{\mathbf{u}}^{(0)}(\psi_1), V_{\mathbf{u}}^{(L)}(\psi_2)]\chi \in O_{\mathbf{u}}, \quad (21)$$

and that for $L \geq 0$,

$$V_{\mathbf{u}}^{(0)} \left(V_{(\infty, -1)}^{(L)}(\psi) \chi \right) \phi \approx V_{\mathbf{u}}^{(L)}(\psi) V_{\mathbf{u}}^{(0)}(\chi) \phi, \quad (22)$$

where \approx denotes equality up to states in $O_{\mathbf{u}}$. This implies that ρ_1 defines an algebra homomorphism $\mathcal{A} \rightarrow \text{End}(A_{\mathbf{n}})$. Using the Möbius invariance of the amplitudes, this is sufficient to prove the statement for all i . The formula (20) follows easily from the definition of the action. \blacksquare

If \mathcal{A} is semisimple and if all u_i are distinct, Theorem 4 allows us to decompose $A_{(u_1, \dots, u_k)}^*$ completely into representations $(R_1 \otimes \dots \otimes R_k)$ of \mathcal{A}^k ; the multiplicity with which $(R_1 \otimes \dots \otimes R_k)$ appears in $A_{(u_1, \dots, u_k)}^*$ then gives an upper bound on the number of different ways in which the spaces $\mathcal{H}^{R_1}, \dots, \mathcal{H}^{R_k}$ can be coupled.[¶]

Given a representation, a rough measure of its size relative to the vacuum representation is given by the *special subspace*, defined by Nahm in [28] as follows: let $W \subset \mathcal{H}^i$ be defined by

$$W = \text{Span}\{V_n(\psi)\chi : n \leq -h_\psi < 0, \psi \in \mathcal{H}, \chi \in \mathcal{H}^i\}. \quad (23)$$

Then a special subspace, \mathcal{H}_s^i , is a subspace of \mathcal{H}^i such that $W + \mathcal{H}_s^i = \mathcal{H}^i$ and $W \cap \mathcal{H}_s^i = \{0\}$. The dimension of \mathcal{H}_s^i equals the dimension of the quotient space \mathcal{H}^i/W , and thus is independent of the choice of \mathcal{H}_s^i . In the case of the vacuum representation, $\dim \mathcal{H}_s = 1$, and $\dim \mathcal{H}_s^i > 1$ for any other representation. Representations whose special subspace is finite-dimensional play a preferred role (their fusion rules are finite), and are called *quasirational*.

Finally, since \mathcal{H}^i carries an action of the $V_n(\psi)$, we note that we can define various quotients $A_{\mathbf{n}}^i$ of \mathcal{H}^i just by replacing \mathcal{H} with \mathcal{H}^i in (7), (8). In particular, when $\mathcal{H}^i = \mathcal{H}^R$, $A_{[1]}^R$ is isomorphic to the highest weight space R , as can be seen from choosing $\mathbf{u} = (\infty)$ in (7). The $A_{\mathbf{n}}^R$ obey an analogue of Theorem 1, but we will not use this fact explicitly in what follows.

5 Rationality

One of the central concepts in conformal field theory is ‘rationality,’ a condition which is supposed to express a kind of finiteness of the theory. There exist various notions of finiteness in the literature [6, 23, 26, 33] and the precise interrelations between the different assumptions are not all understood. On the other hand, most people would agree that every rational theory should have the following properties:

- (i) The conformal field theory has only finitely many irreducible highest weight representations.
- (ii) The characters $\chi_R(q) = \text{Tr}_{\mathcal{H}^R} q^{L_0}$ are convergent for $|q| < 1$ and close under modular transformations.
- (iii) The fusion rule coefficients N_{ij}^k of three irreducible highest weight representations, \mathcal{H}^i , \mathcal{H}^j and \mathcal{H}^k are all finite.

There are various different conditions that imply some of these properties. For example, if Zhu’s algebra is semisimple, it follows from the Wedderburn structure theorem (see for example [9]) that

$$\mathcal{A} = \bigoplus_i \text{End} V_i \quad (24)$$

[¶]For theories in which every representation is completely reducible (see Section 5) this bound is sharp, *i.e.* every element of $A_{\mathbf{u}}^*$ corresponds to an actual coupling. The reason this is not true in general is that the correlation functions coming from an element of $A_{\mathbf{u}}^*$ need not respect the null-vector relations in the \mathcal{H}^{R_i} .

for a finite set of finite-dimensional vector spaces V_i , which form the only irreducible representations of \mathcal{A} . Thus if \mathcal{A} is semisimple, (i) is satisfied. It is reasonable to conjecture that (ii) and (iii) should also follow from the semisimplicity of \mathcal{A} , but this conjecture is, at least at present, still out of reach. In order to make progress, two other conditions have been proposed:

- (a) Every \mathbb{N} -graded weak module is completely reducible. (This is the condition called rationality by Zhu and many other authors on vertex operator algebras [6, 23, 33].)
- (b) The quotient space $A_{[2]}$ is finite-dimensional. (This is the C_2 condition of Zhu.)

It has been shown in [33] that (a) implies the semisimplicity of \mathcal{A} , and therefore by the above argument (i). In the same paper it was shown that (a) together with (b) imply (ii). Zhu further conjectured that (a) implies (b), but this also seems at present out of reach. The C_2 condition implies that \mathcal{A} is finite-dimensional, but does not imply its semisimplicity [16].

In the following we shall mainly analyse the implications of (b). In particular we shall show that (b) implies that every highest weight representation is quasirational and that (iii) holds. We shall also give a direct argument for the convergence of the characters under the assumption of (b).

6 The basis lemma

First we will prove three computational results that are originally due to Borcherds [2] (see also [21]).

Lemma 5. *We have*

$$[V_{(-N_1)}(\psi_1), V_{(-N_2)}(\psi_2)] = \sum_{r=1}^{h_1+h_2} V_{(-N_1-N_2+1-r)}(\chi_r), \quad (25)$$

where h_i is the conformal weight of ψ_i , and the conformal weight of χ_r is $h_1 + h_2 - r$.

Proof. The commutator

$$[V_{-N_1+1-h_1}(\psi_1), V_{-N_2+1-h_2}(\psi_2)] = \sum_{s=0}^{h_1+h_2-1} V_{-N_1-N_2+2-h_1-h_2}(\chi_s), \quad (26)$$

where the conformal weight of χ_s is $h_1 + h_2 - 1 - s$. Substituting $r = s + 1$, we then obtain the above formula. ■

Lemma 6. *We have*

$$\begin{aligned} V_{(-N_1)}(V_{(-N_2)}(\psi)\chi) &= \sum_{L \geq 0} \binom{N_2 + L - 1}{L} V_{(-N_2-L)}(\psi) V_{(-N_1+L)}(\chi) \\ &\quad + (-1)^{N_2+1} \sum_{L \geq 0} \binom{N_2 + L - 1}{L} V_{(-N_1-N_2-L)}(\chi) V_{(L)}(\psi) \end{aligned} \quad (27)$$

where both sums terminate when they are evaluated on an element of \mathcal{H} .

Proof. We rewrite $V_{(-N_1)}(V_{(-N_2)}(\psi)\chi)$ as

$$\begin{aligned} V_{(-N_1)}(V_{(-N_2)}(\psi)\chi) &= \oint_0 V(V_{(-N_2)}(\psi)\chi, \zeta) \zeta^{-N_1} d\zeta \\ &= \oint \oint_{|\zeta| > |z|} V(V(\psi, z)\chi, \zeta) z^{-N_2} \zeta^{-N_1} dz d\zeta \\ &= \oint \oint_{|\zeta| > |z|} V(\psi, z + \zeta) V(\chi, \zeta) z^{-N_2} \zeta^{-N_1} dz d\zeta. \end{aligned} \quad (28)$$

We then substitute $\omega = z + \zeta$ and find

$$\begin{aligned} V_{(-N_1)}(V_{(-N_2)}(\psi)\chi) &= \oint_0 \left\{ \oint_{\zeta} V(\psi, \omega) V(\chi, \zeta) (\omega - \zeta)^{-N_2} d\omega \right\} \zeta^{-N_1} d\zeta \\ &= \oint \oint_{|\omega| > |\zeta|} V(\psi, \omega) \left\{ V(\chi, \zeta) (\omega - \zeta)^{-N_2} \zeta^{-N_1} d\zeta \right\} d\omega \\ &\quad - \oint \oint_{|\zeta| > |\omega|} V(\chi, \zeta) \left\{ V(\psi, \omega) (\omega - \zeta)^{-N_2} d\omega \right\} \zeta^{-N_1} d\zeta. \end{aligned} \quad (29)$$

In the first line we can then write

$$(\omega - \zeta)^{-N_2} = \omega^{-N_2} \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} \left(\frac{\zeta}{\omega} \right)^L,$$

and thus obtain

$$\begin{aligned} &= \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} \oint_0 V(\psi, \omega) \omega^{-N_2-L} d\omega \oint_0 V(\chi, \zeta) \zeta^{-N_1+L} d\zeta \\ &= \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} V_{(-N_2-L)}(\psi) V_{(-N_1+L)}(\chi). \end{aligned} \quad (30)$$

Finally, we rewrite the second line as

$$(\omega - \zeta)^{-N_2} = (-1)^{N_2} \zeta^{-N_2} \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} \left(\frac{\omega}{\zeta} \right)^L,$$

and obtain

$$\begin{aligned} &= (-1)^{N_2+1} \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} \oint_0 V(\chi, \zeta) \zeta^{-N_1-N_2-L} d\zeta \oint_0 V(\psi, \omega) \omega^L d\omega \\ &= (-1)^{N_2+1} \sum_{L=0}^{\infty} \binom{N_2 + L - 1}{L} V_{(-N_1-N_2-L)}(\chi) V_{(L)}(\psi). \end{aligned} \quad (31)$$

This proves the claim. ■

Lemma 7. *As an immediate corollary of Lemma 6, we have*

$$\begin{aligned} V_{(-N)}(\psi) V_{(-N)}(\chi) &= V_{(-2N+1)}(V_{(-1)}(\psi)\chi) - \sum_{L \geq 0, L \neq N} V_{(-1-L)}(\psi) V_{(-2N+1+L)}(\chi) \\ &\quad - \sum_{M \geq 0} V_{(-2N-M)}(\chi) V_{(M)}(\psi), \end{aligned} \quad (32)$$

where again both sums terminate when they are evaluated on an element of \mathcal{H} .

Proof. This follows from Lemma 6 with $N_1 = 2N - 1$ and $N_2 = 1$. ■

The next proposition is the core of this section. Recall that $A_{[2]} \simeq \mathcal{H}/O_{(\infty, \infty)}$ and that $O_{(\infty, \infty)}$ is spanned by states of the form $V_{(-M)}(\rho)\chi$ where $\rho, \chi \in \mathcal{H}$ and $M > 1$.

Proposition 8. *Let $\{W_i\}$ be a set of representatives for \mathcal{H} modulo $O_{(\infty, \infty)}$. Then \mathcal{H} is spanned by the set of states*

$$V_{(-N_1)}(W_{i_1}) \cdots V_{(-N_n)}(W_{i_n})\Omega, \quad (33)$$

where $N_1 > N_2 > \cdots > N_n > 0$.

Proof. Define a filtration on \mathcal{H} ,

$$\mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \cdots \subset \mathcal{H}^{(g)} \subset \cdots \subset \mathcal{H}, \quad (34)$$

as follows: $\mathcal{H}^{(g)}$ is the subspace spanned by all states of the form

$$V_{(-N_1)}(\psi_1) \cdots V_{(-N_n)}(\psi_n)\Omega \quad (35)$$

where $\sum_i h_{\psi_i} \leq g$. Clearly $\mathcal{H} = \cup_g \mathcal{H}^{(g)}$ (since every Ψ has at least the trivial representation $\Psi = V_{(-1)}(\Psi)\Omega$, so that if Ψ is homogeneous we have $\Psi \in \mathcal{H}^{(h_\Psi)}$.)

Two properties of this filtration will be useful in what follows. First, commutator terms always have lower grade: more precisely, let $\Psi \in \mathcal{H}$ be some state of the form (35), with $\sum_i h_{\psi_i} \leq g$, and let Ψ_R be the state obtained from Ψ by exchanging two adjacent modes in (35). Then $\Psi - \Psi_R \in \mathcal{H}^{(g-1)}$, as follows readily from Lemma 5. Second, elements of $O_{(\infty, \infty)}$ decrease the grade: again let $\Psi \in \mathcal{H}$ be of the form (35), with $\sum_i h_{\psi_i} \leq g$, but this time with the additional stipulation that some $\psi_i \in O_{(\infty, \infty)}$, i.e. $\psi_i = V_{(-M)}(\rho)\chi$, $M > 1$. Then using Lemma 6 we find that $\Psi \in \mathcal{H}^{(g-1)}$, since the state $V_{(-M)}(\rho)\chi$ is of weight $h_\chi + h_\rho + (M - 1)$.

For any pair (g, N) of nonnegative integers we now consider the proposition:

Inductive hypothesis. *The space $\mathcal{H}^{(g)}$ is spanned by states of the form*

$$V_{(-N_1)}(W_{i_1}) \cdots V_{(-N_n)}(W_{i_n})\Omega \quad (36)$$

where $N_1 \geq N_2 \geq \cdots \geq N_n > 0$, $\sum_j h_{W_{i_j}} \leq g$, and $N_i = N_{i+1}$ is allowed only for $N_i > N$.

We consider pairs to be ordered lexicographically: so $(g, N) < (g', N')$ if either $g < g'$, or $g = g'$ and $N < N'$. Then the set of pairs is well ordered (every non-empty subset has a smallest member). So we can proceed by induction: fixing (g, N) we assume the hypothesis holds for all smaller pairs and establish it for (g, N) .

In particular, the inductive hypothesis means the proposition is true for $(g - 1, N)$ so that every $\Psi \in \mathcal{H}^{(g-1)}$ can be expressed in the claimed form (this is true even for $g = 0$ since in that case $\mathcal{H}^{(g-1)} = 0$.) As remarked above, provided we begin with monomials (35) with $\sum h_{\psi_i} \leq g$, commutator terms and terms involving states in $O_{(\infty, \infty)}$ will always be in $\mathcal{H}^{(g-1)}$; so in trying to reduce some state (35) with $\sum h_{\psi_i} \leq g$ to the claimed form we are always free to reorder modes and to replace any $V_{(M)}(\psi)$ by $V_{(M)}(W)$ (here and below, we suppress the index on W_i , which plays no role.)

We consider separately the pairs (g, N) with $N = 0$. In this case, given an element of $\mathcal{H}^{(g)}$ of the form (35), we can put it in the claimed form simply by reordering modes into descending order and replacing all ψ_i by W . (If any mode $V_{(M)}(W)$ with $M \geq 0$ appears, it will annihilate the vacuum after the reordering.)

Now suppose $N > 0$ and consider Ψ of the form (35) with $\sum h_{\psi_i} \leq g$. Using the inductive hypothesis applied to $(g, N-1)$ we can write Ψ as a sum of states of the form

$$V_{(-M_1)}(W) \cdots V_{(-M_m)}(W) [V_{(-N)}(W)]^s V_{(-L_1)}(W) \cdots V_{(-L_l)}(W) \Omega, \quad (37)$$

where $M_1 \geq \cdots \geq M_m > N > L_1 > \cdots > L_l \geq 0$, $s \geq 0$. If $s < 2$ then (37) is already a state of the desired sort. If $m \neq 0$ then the expression $[V_{(-N)}(W)]^s \cdots \Omega$ is in $\mathcal{H}^{(g-1)}$ and we can use the inductive hypothesis applied to $(g-1, N)$ to replace it, obtaining a sum of expressions which have no repeated indices at or below N . On the other hand, if $m = 0$ and $s \geq 2$ then we use Lemma 7 to replace the initial pair $V_{(-N)}(W)V_{(-N)}(W)$. This replacement generates two sorts of terms: first, it generates

$$V_{(-2N+1)}(\psi) [V_{(-N)}(W)]^{s-2} V_{(-L_1)}(W) \cdots V_{(-L_l)}(W) \Omega, \quad (38)$$

second, it generates

$$V_{(-N-K)}(\psi) V_{(-N+K)}(\chi) [V_{(-N)}(W)]^{s-2} V_{(-L_1)}(W) \cdots V_{(-L_l)}(W) \Omega, \quad (39)$$

where $K > 0$ (using our freedom to reorder modes.) As usual we are free to replace ψ, χ by W everywhere. Now omitting the first mode from (38) or (39) produces a state $\Psi' \in \mathcal{H}^{(g-1)}$ (unless the first W is actually the vacuum, which can happen in (38) in the special case $N = 1$ — we treat this case separately below). Using the inductive hypothesis for $(g-1, N)$ we then rewrite Ψ' in terms of monomials (36) with no repeated indices at or below N . This yields the desired result, since $2N-1$ and $N+K$ are both greater than N , so that re-attaching the omitted mode does not generate a repeat at or below N .

It only remains to consider (38) in the special case $N = 1$. In this case we can rewrite that term simply as $[V_{(-1)}(W)]^{s-1} \Omega$, and repeat the process until we are left with $V_{(-1)}(W) \Omega$. This completes the proof of the inductive hypothesis for all (g, N) .

To complete the proof of the proposition we use the fact that \mathcal{H} is graded by conformal weight, $\mathcal{H} = \cup_{h \geq 0} \mathcal{H}_h$, where \mathcal{H}_h consists of states of weight h . It is therefore sufficient to show that each \mathcal{H}_h is spanned by states (33) with $N_1 > N_2 > \cdots > N_n > 0$. But this follows directly from the inductive hypothesis together with the fact that the conformal weight of the state in (36) is greater than or equal to $\sum_j (N_j - 1)$; thus if (36) is of weight h , none of the N_i can be greater than $h+1$, so the result follows by choosing $N = h+1$ and sufficiently large g in the inductive hypothesis. This completes the proof. \blacksquare

We remark that the spanning set given by Proposition 8 is not actually a basis; this can be seen already for the minimal model with $c = -22/5$, for which the set $\{W_i\}$ can be taken to be $\{\Omega, L_{-2}\Omega\}$. Then (33) includes both $L_{-3}L_{-2}\Omega$ and $L_{-5}\Omega$, but in fact these two states are linearly dependent. Nevertheless, Proposition 8 is a very useful tool as we shall see momentarily.

Most of the known conformal field theories are generated by a finite set of quasiprimary fields, and are indeed what is called *finite W-algebras*. More precisely, a vertex operator algebra is a finite W -algebra if it contains a finite set of states $W_i \in \mathcal{H}$, $i = 1, \dots, n$, such that \mathcal{H} is spanned by states of the form (36) where $N_1 \geq N_2 \geq \cdots \geq N_n > 0$ and $i_j \geq i_{j+1}$ whenever $N_j = N_{j+1}$. It now follows directly from Proposition 8 that

Corollary 9. *If $A_{[2]}$ is finite-dimensional, then the vertex operator algebra is a finite W -algebra.*

Proof. We take the $\{W_i\} \in \mathcal{H}$ to be a set of representatives for \mathcal{H} modulo $O_{(\infty, \infty)}$ and apply Proposition 8. \blacksquare

It is sometimes assumed in the definition of a vertex operator algebra that each L_0 eigenspace is finite-dimensional. It now follows directly from Proposition 8 that this is automatic provided that $A_{[2]}$ is finite-dimensional.

Actually, Corollary 9 has been proven before in [23]. The generating set Li used was somewhat different, however. He defined a space $C_1 \subset \mathcal{H}$ and then showed that \mathcal{H} is spanned by all states $V_{n_1}(\psi_1) \cdots V_{n_m}(\psi_m)\Omega$, where the ψ_i range over some complementary subspace to C_1 . This result was then refined in [22] where it was observed that the modes can actually be taken in a fixed lexicographical order; furthermore it was shown that \mathcal{H}/C_1 is a “minimal” generating set in a certain sense. These results are actually stronger than our Corollary 9 because finite-dimensionality of \mathcal{H}/C_1 is much weaker than our hypothesis. On the other hand, our spanning set has the significant advantage that it allows us to prove the “no repeat” condition of Proposition 8, which will be critical in the arguments of Sections 7 and 8.

The next result has also been obtained before, in [7]:

Proposition 10. *If $A_{[2]}$ is finite-dimensional then the character*

$$\chi(q) = \text{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}}, \quad (40)$$

which is defined as a formal power series, converges for $0 < |q| < 1$.

Proof. Let us denote by $Q(n, k)$ the number of partitions of n into integers of k colours, with no integer appearing twice in the same colour. Then Proposition 8 implies that

$$\text{Tr}_{\mathcal{H}} q^{L_0} \leq \sum_{n \geq 0} q^n Q(n, k) = \prod_{n > 0} (1 + q^n)^k, \quad (41)$$

where the inequality holds for each coefficient of the power series and hence for real positive q . (We set $k = \dim A_{[2]} - 1$ rather than $k = \dim A_{[2]}$ because we can always choose one of the W_i to be Ω , and $V_{(-N)}(\Omega) = \delta_{N,1} \mathbf{1}$.) The right-hand-side converges for $0 < |q| < 1$ since the modulus of its logarithm is bounded by

$$k \sum_{n=1}^{\infty} |\log(1 + q^n)| \leq k \sum_{n=1}^{\infty} \frac{|q|^n}{(1 - |q|^n)} \leq \frac{k}{(1 - |q|)} \sum_{n=1}^{\infty} |q|^n. \quad (42)$$

By the comparison test this then implies the convergence of the character $\chi(q)$ for $0 < |q| < 1$. ■

We remark that by similar techniques to those used in the proof of Proposition 8 one can show that \mathcal{H}^R is spanned by the states of the form (see also [22] for a similar argument)

$$V_{-N_1}(W_{i_1}) \cdots V_{-N_n}(W_{i_n}) U_i, \quad (43)$$

where U_i runs over a basis of the highest weight space R of \mathcal{H}^R , and $N_1 \geq N_2 \geq \cdots \geq N_n > 0$. If the representation in question is irreducible, $\dim A_{[2]} < \infty$ implies that R is finite-dimensional, and we can bound the character of the representation \mathcal{H}^R (defined in analogy to (40)) by

$$\chi_R(q) \leq (\dim R) q^{-\frac{c}{24}} \left(\prod_{n=1}^{\infty} (1 - q^n) \right)^{-k}. \quad (44)$$

This is again sufficient to prove the convergence of these characters for $0 < |q| < 1$.

7 Nahm's conjecture

In this section we will be exploring some further consequences of the assumption that $A_{[2]}$ is finite-dimensional. We remark that results similar to those appearing in this section have been proven in [23], under the assumption that L_0 acts semisimply on all weak modules. This assumption is somewhat difficult to check in practice, however, and in any case is strictly stronger than finite-dimensionality of $A_{[2]}$.^{||}

We shall first prove that every conformal field theory for which $A_{[2]}$ is finite-dimensional possesses only finitely many n -point functions. Given Theorem 1 this statement follows from the following observation.

Theorem 11. *Suppose $A_{[2]}$ is finite-dimensional. Then all $A_{\mathbf{u}}$ are finite-dimensional.*

Proof. By Lemma 3 we see that it is sufficient to show that all $A_{(\infty^k)}$ are finite-dimensional. By definition,

$$O_{(\infty^k)} = \text{Span}\{V_{(-M)}(\rho)\chi : \rho \in \mathcal{H}, \chi \in \mathcal{H}, M > (k-2)h_\rho + 1\}. \quad (45)$$

Now consider the spanning set for \mathcal{H} provided by Proposition 8. Since $A_{[2]}$ is assumed finite-dimensional we can choose the set $\{W_i\}$ to be finite. So \mathcal{H} is spanned by monomials

$$V_{(-N_1)}(W_{i_1}) \cdots V_{(-N_n)}(W_{i_n})\Omega, \quad (46)$$

where $N_1 > \cdots > N_n > 0$. But if $N_1 > (k-2)\max\{h_{W_i}\} + 1$ then the state (46) is in $O_{(\infty^k)}$. This leaves us only finitely many choices for the N_i , which gives a finite spanning set for $\mathcal{H}/O_{(\infty^k)}$, completing the proof. ■

Now we are in a position to prove Nahm's conjecture. Let \mathcal{H}^R be some irreducible highest weight representation of the conformal field theory. In [28] Nahm defined the special subspace \mathcal{H}_s^R (as discussed in Section 4) and defined \mathcal{H}^R to be *quasirational* if \mathcal{H}_s^R is finite-dimensional. Nahm conjectured that the rationality of the theory implies that all irreducible representations are quasirational. We shall now prove this statement under the condition that $A_{[2]}$ is finite-dimensional.

In fact, we shall prove a slightly stronger statement, namely that all quotient spaces $A_{[n]}^R$ are finite-dimensional. This implies that all representations are quasirational since $\dim A_{[2]}^R \geq \dim \mathcal{H}_s^R$, because

$$A_{[2]}^R \simeq A_{(\infty, \infty)}^R = \mathcal{H}^R / \text{Span}\{V_n(\psi)\chi : n < -h_\psi, \psi \in V, \chi \in \mathcal{H}^R\}. \quad (47)$$

The motivation for our proof comes from the interpretation of the quotients $A_{\mathbf{n}}$ as spaces of correlation functions. From Theorem 11 and Lemma 3 we know that $A_{[2]}$ finite-dimensional implies $A_{[p,1]}$ finite-dimensional for all $p \geq 1$; and from Theorem 1 we know that $A_{[p,1]}^*$ can be understood as the space of correlation functions $\langle \cdots \rangle_\eta$ with the property that

$$\text{div} \langle V(\psi, z) \rangle_\eta dz^{\otimes h_\psi} \geq -ph_\psi[u_1] - h_\psi[u_2]. \quad (48)$$

But this analytic structure is exactly what we would expect from correlation functions that are induced by a single highest weight state at u_2 and a state at u_1 that is annihilated by all $V_n(\psi)$ with $n > (p-1)h_\psi$. If we choose $u_1 = \infty$, $u_2 = 0$, the state at $u_1 = \infty$ defines a linear functional on

^{||}The triplet algebra [16] satisfies the C_2 condition, but it possesses representations for which L_0 does not act semisimply.

the Fock space \mathcal{H}^R at $u_2 = 0$. The property that the state at u_1 is annihilated by the modes with $n > (p-1)h_\psi$ implies then that this functional vanishes on $O_{(\infty^p)}^R \subset \mathcal{H}^R$, and therefore defines a functional on $A_{(\infty^p)}^R$. We therefore expect that we can construct an element of $A_{[p,1]}^*$ from a highest weight state U in the representation R , and an element $\eta \in (A_{[p]}^R)^*$; more specifically, if we evaluate the linear functional in $A_{[p,1]}^*$ on $\chi \in \mathcal{H}$ (now regarding \mathcal{H} as being placed at $1 \in \mathbb{P}$) we should have

$$\langle \eta(\infty) \chi(1) U(0) \rangle = \sum_{n \in \mathbb{Z}} \langle \eta(\infty) (V_{-n}(\chi) U)(0) \rangle \quad (49)$$

$$= \sum_{n=0}^{(p-1)h_\chi} \langle \eta(\infty) (V_{-n}(\chi) U)(0) \rangle, \quad (50)$$

where the terms with $n < 0$ are cut off by the highest weight property of U and the terms with $n > (p-1)h_\chi$ are cut off by the assumption that η vanishes on $O_{(\infty^p)}^R$. This formula motivates the proof of:

Lemma 12. *Let \mathcal{H}^R be any representation of the conformal field theory that is generated from a highest weight state U . Then there is an injection*

$$\sigma : (A_{[p]}^R)^* \hookrightarrow (A_{[p,1]})^*. \quad (51)$$

Proof. We realize $A_{[p]}^R$ as $A_{(\infty^p)}^R$ and $A_{[p,1]}$ as $A_{(\infty^p, -1)}$. Then define σ , as suggested above, by the formula

$$[\sigma(\eta)](\chi) = \eta(V(\chi, 1)U) = \sum_{n=-(p-1)h_\chi}^0 \eta(V_n(\chi)U). \quad (52)$$

In order to check that $\sigma(\eta)$ annihilates $O_{(\infty^p, -1)}$, we observe from (7) that $O_{(\infty^p, -1)}$ is generated by the states of the form $V_{(\infty^p, -1)}^{(M)}(\psi)\chi$, where $M > 0$ and

$$V_{(\infty^p, -1)}^{(M)}(\psi) = \oint_0 \frac{d\zeta}{\zeta^{M+1}} V \left[\left(\frac{(\zeta+1)}{\zeta^{p-1}} \right)^{L_0} \psi, \zeta \right]. \quad (53)$$

It is therefore sufficient to show that for $M > 0$,

$$\eta \left(V(V_{(\infty^p, -1)}^{(M)}(\psi)\chi, 1)U \right) = 0, \quad (54)$$

provided that $\eta \in (A_{(\infty^p)}^R)^*$. Expanding out (53) in terms of modes we have

$$V(V_{(\infty^p, -1)}^{(M)}(\psi)\chi, 1) = \sum_{s=0}^{h_\psi} \binom{h_\psi}{s} V(V_{(-(p-1)h_\psi+s-M-1)}(\psi)\chi, 1). \quad (55)$$

Since the vertex operator is evaluated at $z = 1$, we can rewrite it in terms of a sum over all modes $V_{(r)}(\cdot)$. We then collect together all those terms that have the same conformal weight: this amounts to choosing r (as a function of s) as $r = ph_\psi + h_\chi + M - s + t$, where now t labels the different values for the conformal weight of the resulting state. We then apply Lemma 6 to $V_{(h_\chi+ph_\psi+M-s+t)}(V_{(-(p-1)h_\psi+s-M-1)}(\psi)\chi)$. The first sum contains only terms of the form

$V_{(-R)}(\psi)\phi$ with $R \geq (p-1)h_\psi - s + M + 1$, for which η vanishes by assumption. The second sum gives rise to

$$(-1)^{M+(p-1)h_\psi} \sum_{s=0}^{h_\psi} (-1)^s \binom{h_\psi}{s} \sum_{L \geq 0} \binom{(p-1)h_\psi + M - s + L}{L} V_{(h_\psi + h_\chi - 1 + t - L)}(\chi) V_{(L)}(\psi). \quad (56)$$

All terms with $L \geq h_\psi$ vanish since $V_{(L)}(\psi)U = 0$ as U is a highest weight state. It therefore only remains to check that all the other terms vanish, *i.e.* that

$$\sum_{s=0}^{h_\psi} (-1)^s \binom{h_\psi}{s} \binom{(p-1)h_\psi + M - s + L}{L} = 0 \quad \text{for } L = 0, \dots, h_\psi - 1. \quad (57)$$

In order to prove this identity, we observe that

$$\begin{aligned} & \sum_{s=0}^{h_\psi} (-1)^s \binom{h_\psi}{s} \sum_{L \geq 0} \binom{(p-1)h_\psi + M - s + L}{L} u^L \\ &= \sum_{s=0}^{h_\psi} (-1)^s \binom{h_\psi}{s} \frac{1}{(1-u)^{(p-1)h_\psi + M - s + 1}} \\ &= \frac{1}{(1-u)^{(p-1)h_\psi + M + 1}} \sum_{s=0}^{h_\psi} (-1)^s \binom{h_\psi}{s} (1-u)^s \\ &= \frac{u^{h_\psi}}{(1-u)^{(p-1)h_\psi + M + 1}}. \end{aligned} \quad (58)$$

Thus the left-hand-side of (58) does not have any powers of u below h_ψ , and therefore (57) holds.

To complete the proof we must check that σ is injective, *i.e.* that $\sigma(\eta) = 0$ implies $\eta = 0$. By Theorem 1, $\sigma(\eta) = 0$ means that $\langle \prod_j V(\psi_j, z_j) \rangle_{\sigma(\eta)} = 0$ for all ψ_j and z_j ; and since we can generate any mode acting on U by taking suitable contour integrals of vertex operators, it follows that η annihilates any state generated from U . But since U generates the whole of \mathcal{H}^R this implies that $\eta = 0$. This completes the proof. \blacksquare

Combining Lemma 12 and Theorem 11 we now obtain the desired result:

Theorem 13. *Suppose $A_{[2]}$ is finite-dimensional. Then every irreducible highest weight representation of the conformal field theory is quasirational.*

Proof. Using Theorem 11 and Lemma 3 we see that $A_{[p,1]}$ is finite-dimensional for any $p \geq 1$. Then from Lemma 12 it follows that each $A_{[p]}^R$ is finite-dimensional, and the case $p = 2$ implies that the special subspaces are finite-dimensional. \blacksquare

Finally we observe that the tools we have developed here also allow us to prove that the C_2 condition implies the finiteness of the fusion rules:

Corollary 14. *Suppose $A_{[2]}$ is finite-dimensional and let \mathcal{H}^{R_i} , \mathcal{H}^{R_j} and \mathcal{H}^{R_k} be three highest weight representations of the conformal field theory. Then the fusion rule coefficient N_{ij}^k is finite.*

Proof. From the perspective of correlation functions what we are claiming is that there are only finitely many ways to couple the three highest weight representations; this follows from the finite-dimensionality of $A_{[1,1,1]}^*$, and hence is a consequence of Theorem 11. On the other hand, there are

also more algebraic approaches to fusion products [11, 24]; in lieu of proving that these approaches are equivalent, we remark that it is known [11, 25] that

$$N_{ij}^k \leq \dim \text{Hom}_{\mathcal{A}}(A_{[1,1]}^{R_i} \otimes_{\mathcal{A}} A_{[1]}^{R_j}, A_{[1]}^{R_k}). \quad (59)$$

Since all spaces involved are finite-dimensional we get the desired result. \blacksquare

8 A bound on the central charge

Up to now we have analysed what follows from the C_2 condition of Zhu. As we have seen, this assumption is already sufficient to prove Nahm's conjecture. If we assume in addition that \mathcal{A} is semisimple, then using Zhu's result about the modular properties of the characters (see (ii) in Section 5) we can derive a bound on the effective central charge of the W -algebra. If c denotes the central charge of the Virasoro algebra, the effective central charge, \tilde{c} , is defined to be $\tilde{c} = c - 24h_{\min}$, where h_{\min} is the smallest conformal weight of any state in any (irreducible) highest weight representation of the theory. We can now prove

Proposition 15. *Suppose $A_{[2]}$ is finite-dimensional and \mathcal{A} is semisimple. Then*

$$\tilde{c} \leq \frac{(\dim A_{[2]} - 1)}{2}. \quad (60)$$

Proof. As in the proof of Proposition 10, let $k = \dim A_{[2]} - 1$, and define

$$f_2(q) = \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n). \quad (61)$$

This notation goes back to [30], although we have deviated slightly from their convention by replacing q^2 with q ; we could also write f_2 in terms of conventional theta functions. In terms of this function we can then rewrite (41) as

$$\text{Tr}_{\mathcal{H}} q^{L_0} \leq 2^{-\frac{k}{2}} q^{-\frac{k}{24}} f_2(q)^k. \quad (62)$$

Here and in the following we shall always assume that $0 < q < 1$.

Next we follow closely an argument from [8], using the modular transformation properties of characters that were proven by Zhu [33]. (As pointed out in [22], that proof actually only required the assumptions of the Proposition; by the way, this is the only place where we use the semisimplicity of \mathcal{A} .) If we write $q = e^{2\pi i\tau}$, and $\tilde{q} = e^{-2\pi i/\tau}$, then we have

$$\chi_0(\tilde{q}) = \sum_R a_R \chi_R(q), \quad (63)$$

where the a_R are some coefficients, χ_0 is the character of the vacuum representation, χ_R is the character of the representation \mathcal{H}^R , and the sum is over all irreducible representations of Zhu's algebra. On the other hand, using the modular transformation properties of f_2 (see for example [30]) and (62) we have

$$\chi_0(\tilde{q}) \leq \tilde{q}^{-\frac{(k+c)}{24}} 2^{-\frac{k}{2}} f_2(\tilde{q})^k \quad (64)$$

$$= \tilde{q}^{-\frac{(k+c)}{24}} 2^{-\frac{k}{2}} f_4(q)^k, \quad (65)$$

where

$$f_4(q) = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}). \quad (66)$$

In the limit $\tau \rightarrow i\infty$ ($q \rightarrow 0, \tilde{q} \rightarrow 1$), (63) implies that

$$\chi_0(\tilde{q}) = q^{h_{\min} - \frac{c}{24}}(a + o(1)), \quad (67)$$

where a may be zero, while from (64) we get

$$\chi_0(\tilde{q}) \leq 2^{-\frac{k}{2}}(q^{-\frac{1}{48}} + O(q))^k = 2^{-\frac{k}{2}}q^{-\frac{k}{48}}(1 + O(q)). \quad (68)$$

Comparing (67) and (68) we get the desired result $\tilde{c} \leq k/2$. ■

Incidentally, this proposition makes it clear that the dimension of $A_{[2]}$ will often be bigger than that of $A_{[1,1]}$. For example, for a self-dual theory we have $\dim A_{[1,1]} = 1$, but the proposition implies that $\dim A_{[2]} \geq 2\tilde{c} + 1$. (For the e_8 theory at level 1, $\tilde{c} = 8$, and thus we have that $\dim A_{[2]} \geq 17$. As a matter of fact, we have checked that $\dim A_{[2]} \geq 4124$.)

To a physicist, the above argument can be explained as follows. Recall that a basis for a theory of k free R fermions is given by the states

$$\psi_{-N_1}^{i_1} \cdots \psi_{-N_n}^{i_n} \Omega, \quad (69)$$

where $N_1 > \cdots > N_n > 0$. Comparing this with (33) one might loosely say that the number of degrees of freedom of our theory is bounded above by the number of degrees of freedom in a theory of k free fermions. The effective central charge measures in essence the number of degrees of freedom; since every free fermion contributes $1/2$, this explains the bound $\tilde{c} \leq k/2$.

The original argument of [8] was very similar to that presented above, except that they began with a spanning set (33) where $N_1 \geq N_2 \geq \cdots \geq N_n$ and repeats *are* allowed. In essence, they were therefore comparing the theory to a theory of m free bosons (where m is the dimension of the generating set). The modular argument then involved the η function (rather than the f_2 function), and the bound they obtained was $\tilde{c} < m$. For theories for which an explicit (small) generating set is known, their bound tends to be stronger than (60), although not even this is the case in general: for the $c = -22/5$ minimal model, our bound is $\tilde{c} \leq 1/2$ while the bound in [8] is $\tilde{c} < 1$; in actual fact $\tilde{c} = 2/5$ for this example. At any rate, Proposition 15 gives a bound on the effective central charge in terms of an intrinsic quantity of the vertex operator algebra that can be easily determined.

9 An interpretation of $A_{[p,1]}$

Finally we would like to give a more precise interpretation of the spaces $A_{[p,1]}$: namely, we show that any correlation function of the type described by $A_{[p,1]}^*$ is in fact obtained by inserting one highest weight state and one state annihilated by all $V_n(\psi)$ with $n > (p-1)h_\psi$. To prove this result, strengthening Lemma 12, we will need to make a rather strong assumption on the theory: namely, we assume that every weak module is completely reducible into irreducible modules (this property has been called *regularity* in the literature on vertex operator algebras; in particular, it was shown in [23] that regularity actually implies $\dim A_{[2]} < \infty$.) Then we can prove

Proposition 16. *Suppose every weak module is completely reducible. Then*

$$\bigoplus_R (A_{[p]}^R)^* \otimes R \simeq A_{[p,1]}^*, \quad (70)$$

where the sum runs over all irreducible representations of Zhu's algebra \mathcal{A} .

Proof. We claim that the isomorphism is implemented by the map

$$[\sigma(\eta \otimes U)](\chi) = \sum_{n=-(p-1)h_\chi}^0 \eta(V_n(\chi)U). \quad (71)$$

The calculation in the proof of Lemma 12 demonstrates that σ , as given in (71), is well defined. In order to prove that σ is injective, we note that the argument in the proof of Lemma 12 shows that $\sigma(\eta \otimes U) = 0$ only if $\eta \otimes U = 0$. Now suppose σ is not injective. Then there exists some linear dependence

$$\sum_{i=1}^m \sigma(\eta_i \otimes U_i) = 0. \quad (72)$$

Choose such a dependence with the smallest possible m ; we have already observed that $m = 1$ is impossible. If $m > 1$ then U_1 and U_2 cannot be linearly dependent (else we could easily reduce m , contradicting the minimality.) The complete reducibility implies that Zhu's algebra is semisimple [33], and (24) then guarantees that there exists some $a \in \mathcal{A}$ with $aU_1 = 0$, $aU_2 \neq 0$; equivalently, there exists some $\psi \in \mathcal{H}$ such that $V_0(\psi)U_1 = 0$, $V_0(\psi)U_2 \neq 0$. Next we use Theorem 1 to identify $A_{(\infty^p, -1)}^*$ with a space of correlation functions. We can therefore re-express (72) as the statement that

$$\sum_{i=1}^m \langle \prod_j V(\psi_j, z_j) \rangle_{\sigma(\eta_i \otimes U_i)} = 0, \quad (73)$$

for all ψ_j and z_j . By taking a suitable contour integral this implies in particular that

$$\sum_{i=1}^m \langle \prod_j V(\psi_j, z_j) V_0(\psi) \rangle_{\sigma(\eta_i \otimes U_i)} = 0, \quad (74)$$

and therefore that

$$\sum_{i=2}^m \sigma(\eta_i \otimes aU_i) = 0, \quad (75)$$

contradicting the minimality of m . This completes the proof of the injectivity.

It remains to show that σ is surjective. Because of Theorem 4 Zhu's algebra \mathcal{A} acts on $A_{(\infty^p, -1)}^*$ via its action at -1 , and we can therefore decompose $A_{(\infty^p, -1)}^*$ as

$$A_{(\infty^p, -1)}^* = \bigoplus_R B_{[p]}^R \otimes R, \quad (76)$$

where $B_{[p]}^R$ denotes an as yet undetermined multiplicity space. Using Theorem 1 we can regard $A_{(\infty^p, -1)}^*$ as the space of correlation functions $\langle \prod_j V(\psi_j, z_j) \rangle_\eta$, satisfying the conditions

$$\langle \prod_j V(\psi_j, z_j) V_n(\psi) \rangle_\eta = 0 \quad \text{for } n > 0, \quad (77)$$

$$\langle V_n(\psi) \prod_j V(\psi_j, z_j) \rangle_\eta = 0 \quad \text{for } n < -(p-1)h_\psi. \quad (78)$$

Using the decomposition (76), $B_{[p]}^R$ can then be regarded as the space of correlation functions for which the zero modes in (77) transform in the representation R of \mathcal{A} . Each element of $B_{[p]}^R$ defines a representation of the conformal field theory where the state at -1 is a highest weight state (that transforms in the representation R under the action of the zero modes), whereas the state at ∞ is only annihilated by the modes $V_n(\psi)$ with $n > (p-1)h_\psi$.

Now we would like to argue that each $\xi \in B_{[p]}^R$ actually defines a linear functional on \mathcal{H}^R , the Fock space generated by the action of the modes on the highest weight state at -1 . We might *a priori* worry that the correlation functions associated with ξ did not respect the null-vector relations by which one quotients in the definition of \mathcal{H}^R ; indeed, in the definition of \mathcal{H}^R we divided out states that vanish in amplitudes involving an arbitrary number of vertex operators and a highest weight state in the (dual) representation, but now we are considering what seem to be more general amplitudes. To resolve this difficulty we use our extra assumption of complete reducibility. The condition (78) is sufficient to deduce that the Fock space that is generated by the action of the modes on the state at ∞ defines a weak module, and therefore must be completely reducible into a direct sum of irreducible highest weight representations. Thus in fact we are only considering amplitudes where, apart from an arbitrary number of vertex operators, we have a highest weight state at ∞ , and therefore $\xi \in B_{[p]}^R$ indeed defines a linear functional on \mathcal{H}^R . It follows from (78) that this functional vanishes on $O_{(\infty^p)}^R$, and hence that it can be regarded as a linear functional on $A_{(\infty^p)}^R \simeq A_{[p]}^R$. It therefore follows that $B_{[p]}^R \simeq (A_{[p]}^R)^*$, and we have thus established the proposition. \blacksquare

Proposition 16 implies in particular that the dimension of the quotient spaces $A_{[p]}^R$ for each representation \mathcal{H}^R is bounded in terms of the quotient space $A_{[p+1]}$ of the vacuum representation. This result reflects the familiar fact that, for rational theories, the vacuum representation already contains a substantial amount of information about all representation spaces \mathcal{H}^R .

10 Conclusions

In this paper we have proven the conjecture of Nahm that every representation of a rational conformal field theory is quasirational (Theorem 13). More specifically, we have shown that if the conformal field theory satisfies the C_2 condition of Zhu, *i.e.* if the space $A_{[2]}$ is finite-dimensional, then the quotient space $A_{[p]}^R$ of each highest weight representation \mathcal{H}^R is finite-dimensional for $p \geq 1$; this immediately implies that \mathcal{H}^R is quasirational. We have also shown that this implies that the theory has only finitely many n -point functions, and in particular that the fusion rules between irreducible representations are finite (Corollary 14). The main technical result of the paper is the spanning set for the vacuum representation of any conformal field theory (Proposition 8), from which we have also been able to deduce various other properties of conformal field theories that satisfy the C_2 condition of Zhu (Corollary 9 and Proposition 10).

We have introduced systematically spaces $A_{\mathbf{u}}$ that describe the correlation functions with k highest weight states at u_1, \dots, u_k . Some of the structure of these spaces does not depend on whether the u_i are pairwise distinct, and one may therefore hope that these spaces will be useful in extending the definition of conformal field theory to singular limits, as envisaged in the program of Friedan & Shenker [13].

In [29] it was shown that the finite-dimensionality of $A_{[n]}$ implies the existence of n -point functions satisfying the Knizhnik-Zamolodchikov equation. Given Theorem 11, it now follows that the existence of n -point functions already follows from the finite-dimensionality of $A_{[2]}$. Similarly,

the condition that $A_{(\infty^k)}$ is finite-dimensional in Theorem 2 can now be relaxed to the assumption that $A_{[2]}$ is finite-dimensional.

It may be possible to prove an inhomogeneous version of the finiteness lemma (Proposition 8). In particular, one may be able to prove that the finite dimensionality of \mathcal{A} implies the finite dimensionality of all $A_{[1,1,\dots,1]}$. This would go a certain way to proving (a version of) Zhu's conjecture, that the finite dimensionality of Zhu's algebra implies that the C_2 condition is satisfied.

However, it seems likely that this will require more sophisticated methods, since the conjecture apparently does not hold for meromorphic field theories (that are not conformal). Consider the theory for which V is spanned by states $J^{a,i}$ of grade 1, where $a = 1, \dots, 248$ labels the adjoint representation of e_8 , and $i \in \mathcal{I}$, where \mathcal{I} is some countably infinite set. For any finite set of vectors in V we can define the amplitudes to be the products of the amplitudes that are associated to the different copies of the affine e_8 theory at level 1. These amplitudes are well defined and satisfy all the conditions of [15] (except that the theory does not have a conformal structure and the weight spaces are not finite-dimensional). Since each e_8 level 1 theory is self-dual, it is easy to see that the same holds for the infinite tensor theory; thus \mathcal{A} is one-dimensional. However, the eigenspace at conformal weight 1 is infinite-dimensional, and Proposition 10 therefore implies that the C_2 condition cannot be satisfied. On the other hand, most of our arguments (in particular all of Section 6 and 7) do not require a conformal structure or the assumption that the L_0 eigenspaces are finite-dimensional.

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A The action of Zhu's algebra on $A_{\mathbf{u}}$

In this appendix we want to prove (21) and (22). Both these statements follow from straightforward calculations.

A.1 Proof of (21)

Without loss of generality we may assume that ψ_i , $i = 1, 2$ are both vectors of definite conformal weight h_i . Using (9), we can then write the commutator $[V_{\mathbf{u}}^{(0)}(\psi_1), V_{\mathbf{u}}^{(L)}(\psi_2)]$ as

$$\begin{aligned}
& \oint_0 \left\{ \oint_{\zeta} V(\psi_1, z) V(\psi_2, \zeta) \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \frac{dz}{z} \right\} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_2} \frac{d\zeta}{\zeta^{L+1}} \\
&= \oint_0 \left\{ \oint_{\zeta} V(V(\psi_1, z - \zeta) \psi_2, \zeta) \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \frac{dz}{z} \right\} \\
&\quad \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_2} \frac{d\zeta}{\zeta^{L+1}} \\
&= \sum_{m=0}^{h_1+h_2-1} \oint_0 V(V_{m+1-h_1}(\psi_1) \psi_2, \zeta) \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_2} \frac{d\zeta}{\zeta^{L+1}} \\
&\quad \left\{ \oint_{\zeta} (z - \zeta)^{-m-1} \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \frac{dz}{z} \right\}. \tag{79}
\end{aligned}$$

The integral in brackets is

$$\begin{aligned}
& \frac{1}{m!} \frac{d^m}{dz^m} \left[\left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \frac{1}{z} \right] \Big|_{z=\zeta} \\
&= \frac{1}{m!} \sum_{s=0}^m \binom{m}{s} (-1)^s \frac{1}{\zeta^{1+s}} \frac{d^{m-s}}{dz^{m-s}} \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \Big|_{z=\zeta}, \tag{80}
\end{aligned}$$

and the last derivative is of the form

$$\frac{d^{m-s}}{dz^{m-s}} \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_1} \Big|_{z=\zeta} = \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_1-m+s} \left[\binom{h_1}{m-s} + O(\zeta^{-1}) \right], \tag{81}$$

where the last bracket consists of a finite sum of terms. Thus (80) becomes

$$\begin{aligned}
& \sum_{s=0}^m C_s \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_1-m-1} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-1}} \right)^{1+s} [1 + O(\zeta^{-1})] \\
&= \sum_{s=0}^m \hat{C}_s \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_1-m-1} [1 + O(\zeta^{-1})], \tag{82}
\end{aligned}$$

where C_s and \hat{C}_s are some constants. Putting this back into (79) and observing that the conformal weight of $V_{m+1-h_1}(\psi_1) \psi_2$ is $h_1 + h_2 - m - 1$, we obtain the statement.

A.2 Proof of (22)

We rewrite the left-hand-side of (22) as

$$\begin{aligned}
&= \oint \frac{d\zeta}{\zeta} \oint \frac{dw}{w^{L+1}} (w+1)^{h_\psi} V \left[\left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{L_0} V(\psi, w) \chi, \zeta \right] \\
&= \oint \frac{d\zeta}{\zeta} \oint \frac{dw}{w^{L+1}} (w+1)^{h_\psi} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_\psi + h_\chi} V \left[V \left(\psi, w \frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right) \chi, \zeta \right] \\
&= \oint_0 \frac{d\zeta}{\zeta} \oint_\zeta \frac{dz}{(z-\zeta)^{L+1}} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_\chi + L} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + (z - \zeta) \right)^{h_\psi} V(\psi, z) V(\chi, \zeta),
\end{aligned} \tag{83}$$

where in the first two lines the integrals are taken over the region $|\zeta| > |w|$, and we have substituted, in the last line,

$$z = w \frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + \zeta.$$

Using the usual contour deformation trick, the last line of (83) can be written as the difference of two contour integrals

$$\begin{aligned}
&= \oint \oint_{|z| > |\zeta|} d\zeta dz \frac{1}{\zeta(z-\zeta)^{L+1}} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_\chi + L} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + (z - \zeta) \right)^{h_\psi} V(\psi, z) V(\chi, \zeta) \\
&\quad - \oint \oint_{|z| > |\zeta|} d\zeta dz \frac{1}{\zeta(z-\zeta)^{L+1}} \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} \right)^{h_\chi + L} \\
&\quad \quad \quad \left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + (z - \zeta) \right)^{h_\psi} V(\chi, \zeta) V(\psi, z).
\end{aligned} \tag{84}$$

The two terms can now be considered separately. In the second term we write

$$\frac{1}{(z-\zeta)^{L+1}} = (-1)^{L+1} \frac{1}{\zeta^{L+1}} \sum_{M=0}^{\infty} \binom{L+M}{M} \left(\frac{z}{\zeta} \right)^M,$$

and observe that

$$\left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + (z - \zeta) \right)^{h_\psi} = z^{h_\psi} \left(1 + \frac{c_1}{z} + \mathcal{O}\left(\frac{z}{\zeta}\right) \right).$$

The second term therefore consists of terms of the form $V_{\mathbf{u}}^{(M)}(\chi) \hat{\phi}$ with $M > 0$, and therefore can be dropped. In reaching this conclusion we have used that if ϕ is in the Fock space, only finitely many powers of $\frac{z}{\zeta}$ contribute.

In the first term we now write

$$\frac{dz}{(z-\zeta)^{L+1}} = \frac{dz}{z^{L+1}} \sum_{M=0}^{\infty} \binom{L+M}{M} \left(\frac{\zeta}{z} \right)^M,$$

and observe that

$$\begin{aligned}
\left(\frac{\prod_{j=2}^k (\zeta - u_j)}{\zeta^{k-2}} + (z - \zeta) \right)^{h_\psi} &= \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_\psi} \left(\frac{z^{k-2}(z - \zeta)}{\prod_{j=2}^k (z - u_j)} + \frac{\prod_{j=2}^k \frac{(\zeta - u_j)}{(z - u_j)}}{\left(\frac{\zeta}{z} \right)^{k-2}} \right)^{h_\psi} \\
&= \left(\frac{\prod_{j=2}^k (z - u_j)}{z^{k-2}} \right)^{h_\psi} \left[1 + \mathcal{O} \left(\frac{\zeta}{z} \right) \right].
\end{aligned} \tag{85}$$

Putting this back into (84) proves (22). Again, we have used here that if ϕ is in the Fock space, only finitely many powers of $\frac{\zeta}{z}$ contribute.

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